

Horoball packings and their densities by generalized simplicial density function in the hyperbolic space *

Jenő Szirmai

Budapest University of Technology and Economics

Institute of Mathematics, Department of Geometry

H-1521 Budapest, Hungary

Email: szirmai@math.bme.hu

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Abstract

The aim of this paper is to determine the locally densest horoball packing arrangements and their densities with respect to fully asymptotic tetrahedra with at least one plane of symmetry in hyperbolic 3-space $\overline{\mathbf{H}}^3$ extended with its absolute figure, where the ideal centers of horoballs give rise to vertices of a fully asymptotic tetrahedron. We allow horoballs of different types at the various vertices. Moreover, we generalize the notion of the simplicial density function in the extended hyperbolic space $\overline{\mathbf{H}}^n$, ($n \geq 2$), and prove that, in this sense, *the well known Böröczky–Florian density upper bound for "congruent horoball" packings of $\overline{\mathbf{H}}^3$ does not remain valid to the fully asymptotic tetrahedra.*

The density of this locally densest packing is ≈ 0.874994 , may be surprisingly larger than the Böröczky–Florian density upper bound ≈ 0.853276 but our local ball arrangement seems not to have extension to the whole hyperbolic space.

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1 Basic notions

1.1 Local density function

We summarize the most important definitions and results about ball packings in $\overline{\mathbf{H}}^n$ ($n \geq 2$). For more details and proofs, we refer to [1], [4], [8], [9] and [16]. There are different notions of packing density. For later purposes, the local density measure is the best suited one. In the n -dimensional ($n \geq 2$) hyperbolic space there are 3-types of spheres: sphere, horosphere and hypersphere. Now, we consider the horospheres and their bodies, the horoballs. A horoball packing $\mathcal{B}_h = \{B_h\}$ of $\overline{\mathbf{H}}^n$ is an arrangement of non-overlapping horoballs B_h in $\overline{\mathbf{H}}^n$. The notion of local density of the usual ball packing can be extended for horoball packings \mathcal{B}_h of $\overline{\mathbf{H}}^n$. Let $B_h \in \mathcal{B}_h$, and $P \in \overline{\mathbf{H}}^n$ an arbitrary point. Then, $\rho(P, B_h)$ is defined to be the length of the unique perpendicular from P to the horosphere S_h bounding B_h , where again $\rho(P, B_h)$ is taken negative for $P \in B_h$. The Dirichlet–Voronoi cell (shortly D-V cell) $\mathcal{D}(B_h)$ of B_h in \mathcal{B}_h is defined to be the convex body

$$\mathcal{D}_h = D(\mathcal{B}_h, B_h) := \{P \in \mathbf{H}^n \mid \rho(P, B_h) \leq \rho(P, B'_h), \forall B'_h \in \mathcal{B}_h\}. \quad (1.1)$$

Since both, B_h and \mathcal{D}_h , are of infinite volume, the usual concept of local density has to be modified. Let $Q \in \partial\mathbf{H}^n$ denote the base point (ideal center at the infinity) of B_h , and interpret S_h as a Euclidean $(n-1)$ -space. Let $B_{n-1}(R) \subset S_h$ be an $n-1$ -ball with center $C \in S_h$. Then, $Q \in \partial\mathbf{H}^n$ and $B_{n-1}(R)$ determine a convex cone $C_n(R) := \text{cone}(B_{n-1}^Q(R)) \in \overline{\mathbf{H}}^n$ with apex Q consisting of all hyperbolic geodesics through $B_{n-1}(R)$ with limiting point Q . With these preparations, the local density $\delta_n(B_h, \mathcal{B}_h)$ of B_h to \mathcal{D}_h is defined by

$$\delta_n(\mathcal{B}_h, B_h) := \lim_{R \rightarrow \infty} \frac{\text{vol}(B_h \cap C_n(R))}{\text{vol}(\mathcal{D}_h \cap C_n(R))}, \quad (1.2)$$

and this limes superior is independent of the choice of the center C of $B_{n-1}(R)$.

1.2 Densest packings with horoballs of the same type

We have to change the notion of the „congruent” horoballs in a horoball packing to the horoballs of the ”same type” because the horoballs are congruent in the hyperbolic space $\overline{\mathbf{H}}^n$. Two horoballs in a horoball packing are in the ”same type” if and only if the local densities of the horoballs to the corresponding cell (e.g.

D-V cell; or ideal simplex, later on) are equal. If we assume that the „horoballs belong to the same type”, then by analytical continuation, the well known simplicial density function on $\overline{\mathbf{H}}^n$ can be extended from n -balls of radius r to the case $r = \infty$, too. Namely, in this case consider $n + 1$ horoballs B_h which are mutually tangent. The convex hull of their base points at infinity will be a totally asymptotic or ideal regular simplex $T_{reg} \in \overline{\mathbf{H}}^n$ of finite volume. Hence, in this case it is legitimate to write

$$d_n(\infty) = (n + 1) \frac{\text{vol}(B_h) \cap T_{reg}}{\text{vol}(T_{reg})}. \quad (1.3)$$

Then for a horoball packing \mathcal{B}_h , there is an analogue of ball packing, namely (cf. [4], Theorem 4)

$$\delta_n(\mathcal{B}_h, B_h) \leq d_n(\infty), \quad \forall B_h \in \mathcal{B}_h. \quad (1.4)$$

The upper bound $d_n(\infty)$ ($n = 2, 3$) is attained for a regular horoball packing, that is, a packing by horoballs which are inscribed in the cells of a regular honeycomb of $\overline{\mathbf{H}}^n$. For dimensions $n = 2$, there is only one such packing. It belongs to the regular tessellation $\{\infty, 3\}$. Its dual $\{3, \infty\}$ is the regular tessellation by ideal triangles all of whose vertices are surrounded by infinitely many triangles. This packing has in-circle density $d_2(\infty) = \frac{3}{\pi} \approx 0.95493\dots$

In $\overline{\mathbf{H}}^3$ there is exactly one horoball packing whose Dirichlet–Voronoi cells give rise to a regular honeycomb described by the Schläfli symbol $\{6, 3, 3\}$. Its dual $\{3, 3, 6\}$ consists of ideal regular simplices T_{reg} with dihedral angle $\frac{\pi}{3}$ building up a 6-cycle around each edge of the tessellation.

1.3 Optimal packings by horoballs of different types in $\overline{\mathbf{H}}^n$

In [9] we have determined the optimal horoball packing arrangements and their densities for all four fully asymptotic Coxeter tilings (Coxeter honeycombs) in $\overline{\mathbf{H}}^3$. Centers of horoballs are required to lie at vertices of the regular polyhedral cells constituting the tiling. **We allow horoballs of different types at the various vertices.** We have proved that the known Böröczky–Florian density upper bound for ”congruent horoball” packings of $\overline{\mathbf{H}}^3$ remains valid for the class of fully asymptotic Coxeter tilings, even if packing conditions are relaxed by allowing horoballs of different types under prescribed symmetry groups. The consequences of this remarkable result are discussed for various Coxeter tilings (see [9]), but in this paper we consider only the tetrahedral Coxeter tilings.

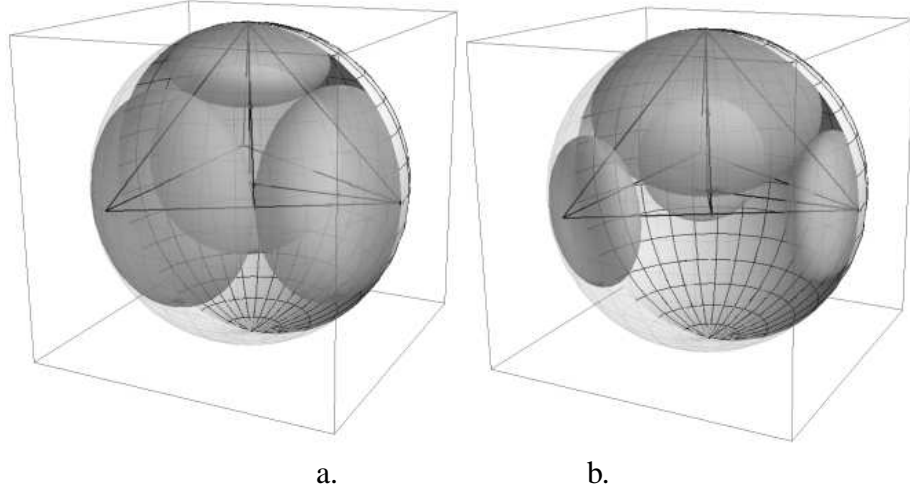


Figure 1: Two optimal horoball arrangements of $(3, 3, 6)$ tiling.

The Coxeter tiling $\{3, 3, 6\}$ is a three-dimensional honeycomb with cells comprised of fully asymptotic regular tetrahedra. We arbitrarily choose such a tetrahedron $T_{reg} = E_0E_1E_2E_3$, and place the horoball centers at the ideal vertices E_0, \dots, E_3 . We vary the types of the horoballs so that they satisfy our constraints of non-overlap. The density of horoball packings to the above honeycomb can be computed by the new type of the simplicial density function:

Definition 1.1 *The density of a horoball packing in Coxeter honeycomb $\{3, 3, 6\}$ is defined as*

$$\delta(\mathcal{B}_{336}) = \frac{\sum_{i=1}^4 \text{vol}(B_i \cap T_{reg})}{\text{vol}(T_{reg})}. \quad (1.5)$$

Theorem 1.2 *There are two distinct optimally dense horoball arrangements \mathcal{B}_{336}^i , ($i = 1, 2$) for the tetrahedral Coxeter tiling $(3, 3, 6)$ with the same density: $\delta(\mathcal{B}_{336}^i) \approx 0.85327609$.*

Fig. 1. shows the two optimal horoball arrangements in Beltrami-Cayley-Klein model.

1.4 Generalization of the simplicial density function

Definition 1.3 We consider an arbitrary fully asymptotic simplex $T = E_0E_1E_2E_3 \dots E_n$ in the n -dimensional hyperbolic space $\overline{\mathbf{H}}^n$. Centers of horoballs are required to lie at vertices of T . We allow horoballs $(B_i, i = 1, 2, \dots, n)$ of different types at the various vertices and require to form a packing, moreover we assume that

$$\text{card}(B_i \cap [E_{i_0}E_{i_1} \dots E_{i_{n-1}}]) \leq 1, \quad i_j \neq i, \quad j \in \{0, 1, \dots, n-1\}.$$

(The hyperplane of points $E_{i_0}, E_{i_1}, \dots, E_{i_{n-1}}$ is denoted by $[E_{i_0}E_{i_1} \dots E_{i_{n-1}}]$ may touch the horoball B_{i_n} .) The generalized simplicial density function for the above simplex and horoballs is defined as

$$\delta(\mathcal{B}) = \frac{\sum_{i=0}^n \text{vol}(B_i \cap T)}{\text{vol}(T)}.$$

For $n = 3$ the main problem is to find a fully asymptotic tetrahedron $T_{\text{opt}} \in \overline{\mathbf{H}}^3$ and horoballs B_i centered at the vertices E_i such that the density $\delta(\mathcal{B})$ (see Definition 1.3) of the corresponding horoball arrangement is maximal. In this case the horoball arrangement \mathcal{B} is said to be *locally optimal*.

In this paper, we study locally optimal horoball packings for fully asymptotic tetrahedra in $\overline{\mathbf{H}}^3$, while allowing different types of horoballs to be centered at the vertices of the fully asymptotic tetrahedra.

The general investigation of this problem seems to be very difficult, thus we assume for simplicity that the tetrahedron has at least one plane of symmetry, simply called plane symmetric. So it will have only one free angle parameter. Then this ideal tetrahedron has 2 orthogonal symmetry planes and 2 symmetry lines of these planes, i.e. the symmetry group is of order 8, denoted by $2 * 2 = \overline{4}m2$ (by Conway and Hermann-Mauguin, respectively).

2 Computations in projective model

For $\overline{\mathbf{H}}^n$ $n \geq 2$ we use the projective model in Lorentz space $\mathbf{E}^{1,n}$ of signature $(1, n)$, i.e. $\mathbf{E}^{1,n}$ is the real vector space \mathbf{V}^{n+1} equipped with the bilinear form of signature $(1, n)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n \quad (2.1)$$

where the non-zero vectors

$$\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{n+1} \text{ and } \mathbf{y} = (y^0, y^1, \dots, y^n) \in \mathbf{V}^{n+1},$$

are determined up to real factors and they represent points in $\mathcal{P}^n(\mathbf{R})$. \mathbf{H}^n is represented as the interior of the absolute quadratic form

$$Q = \{[\mathbf{x}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0\} = \partial \mathbf{H}^n \quad (2.2)$$

in real projective space $\mathcal{P}^n(\mathbf{V}^{n+1}, \mathbf{V}_{n+1})$. All proper interior point $\mathbf{x} \in \mathbf{H}^n$ are characterized by $\langle \mathbf{x}, \mathbf{x} \rangle < 0$.

The points on the boundary $\partial \mathbf{H}^n$ in \mathcal{P}^n represent the absolute points at infinity of $\overline{\mathbf{H}}^n$. Points \mathbf{y} with $\langle \mathbf{y}, \mathbf{y} \rangle > 0$ lie outside of $\overline{\mathbf{H}}^n$ and are called outer points of \mathbf{H}^n . Let $X([\mathbf{x}]) \in \mathcal{P}^n$ a point; $[\mathbf{y}] \in \mathcal{P}^n$ is said to be conjugate to $[\mathbf{x}]$ relative to Q when $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. The set of all points conjugate to $X([\mathbf{x}])$ form a projective polar hyperplane

$$pol(X) := \{[\mathbf{y}] \in \mathcal{P}^n \mid \langle \mathbf{x}, \mathbf{y} \rangle = 0\}. \quad (2.3)$$

Hence the bilinear form Q by (2.1) induces a bijection (linear polarity $\mathbf{V}^{n+1} \rightarrow \mathbf{V}_{n+1}$) from the points of \mathcal{P}^n onto its hyperplanes.

Point $X[\mathbf{x}]$ and the hyperplane $\alpha[\mathbf{a}]$ are called incident if the value of the linear form \mathbf{a} on the vector \mathbf{x} is equal to zero; i.e., $\mathbf{x}\mathbf{a} = 0$ ($\mathbf{x} \in \mathbf{V}^{n+1} \setminus \{0\}$, $\mathbf{a} \in \mathbf{V}_{n+1} \setminus \{0\}$). Straight lines in \mathcal{P}^n are characterized by the 2-subspaces of \mathbf{V}^{n+1} or $(n-1)$ -spaces of \mathbf{V}_{n+1} (see e.g. in [11]).

In this paper we set the sectional curvature of $\overline{\mathbf{H}}^n$, $K = -k^2$, to be $k = 1$. The distance s of two proper points (\mathbf{x}) and (\mathbf{y}) is calculated by the formula:

$$\cosh s = \frac{-\langle \mathbf{x}, \mathbf{y} \rangle}{\sqrt{\langle \mathbf{x}, \mathbf{x} \rangle \langle \mathbf{y}, \mathbf{y} \rangle}}. \quad (2.4)$$

The foot point $Y(\mathbf{y})$ of the perpendicular, dropped from the point $X(\mathbf{x})$ on the plane (u) , has the following form:

$$\mathbf{y} = \mathbf{x} - \frac{\langle \mathbf{x}, \mathbf{u} \rangle}{\langle \mathbf{u}, \mathbf{u} \rangle} \mathbf{u}. \quad (2.5)$$

2.1 On horospheres in hyperbolic space $\overline{\mathbf{H}}^3$

Definition 2.1 *A horosphere in the hyperbolic geometry is the surface orthogonal to the set of parallel lines, passing through the same point on the absolute quadratic surface (simply absolute) of $\overline{\mathbf{H}}^3$.*

We represent $\overline{\mathbf{H}}^3$ by the Beltrami-Cayley-Klein ball model. We introduce a projective coordinate system using vector basis \mathbf{b}_i ($i = 0, 1, 2, 3$) for \mathcal{P}^3 where the coordinates of the center of the model is $(1, 0, 0, 0)$. We pick up an arbitrary point at infinity to be $E_3(1, 0, 0, 1)$ (see Fig. 3).

We obtain the following equation for the horosphere in our Beltrami-Cayley-Klein model of $\overline{\mathbf{H}}^3$ above:

$$-2sx^0x^0 - 2x^3x^3 + 2(s+1)(x^0x^3) + (s-1)(x^1x^1 + x^2x^2) = 0 \quad (2.6)$$

Remark 2.2 1. We get the equation of the horosphere in the Cartesian coordinate system ($x := \frac{x^1}{x^0}$, $y := \frac{x^2}{x^0}$, $z := \frac{x^3}{x^0}$):

$$\frac{2(x^2 + y^2)}{1 - s} + \frac{4(z - \frac{s+1}{2})^2}{(1 - s)^2} = 1. \quad (2.7)$$

2. We will also use the equation of the horosphere with center E_2 (see Fig. 3):

$$\frac{2(\frac{1}{4}x^2 + y^2 + \frac{3}{4}z^2 + \frac{\sqrt{3}}{2}xz)}{1 - s_1} + \frac{4(\frac{\sqrt{3}}{2}x - \frac{1}{2}z - \frac{s_1+1}{2})^2}{(1 - s_1)^2} = 1.$$

The length $l(x)$ of a horocycle arc to a chord segment x is determined by the classical formula due to J. Bolyai:

$$l(x) = k \sinh \frac{x}{k}. \quad (2.8)$$

The intrinsic geometry of the horosphere is Euclidean, therefore, the area \mathcal{A} of a horospherical triangle will be computed by the formula of Heron. The volume of the horoball sectors can be calculated using another formula by J. Bolyai. If the area of a domain on the horosphere is \mathcal{A} , the volume determined by \mathcal{A} and the aggregate of axes drawn from \mathcal{A} is equal to

$$V = \frac{1}{2}k\mathcal{A}, \quad (2.9)$$

we assume that $k = 1$ in this paper.

2.2 The volume of the fully asymptotic tetrahedron in $\overline{\mathbf{H}}^3$

A tetrahedron T in hyperbolic space is determined in general by its six dihedral angles i.e. the mutual angles between the four faces of this tetrahedron. In order to calculate the volume $vol(T)$ of T ; one can dissect T into six orthoschemes and their volumes can be calculated by the Lobachevsky formula.

A plane orthoscheme is a right-angled triangle, whose area can be expressed by the well known defect formula. For three-dimensional spherical orthoschemes, Ludwig Schläfli, about 1850, was able to find the volume differential depending on differentials of the not fixed 3 dihedral angles. Already in 1836, Lobachevsky found a volume formula for three-dimensional hyperbolic orthoschemes \mathcal{O} [3].

The integration method for orthoschemes of dimension three was generalized by Böhm in 1962 [3] to spaces of constant nonvanishing curvature of arbitrary dimension.

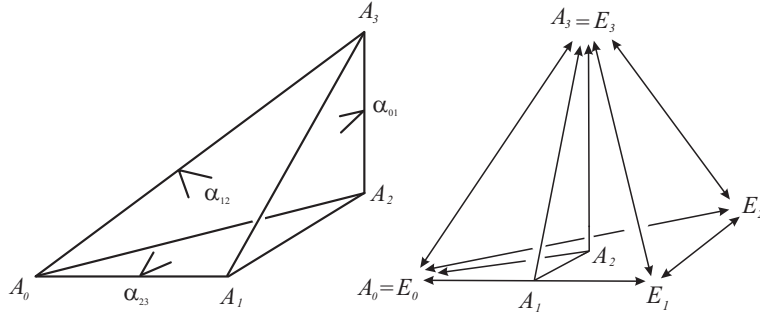


Figure 2:

Theorem 2.3 (N. I. Lobachevsky). *The volume of a three-dimensional hyperbolic orthoscheme $\mathcal{O} \subset \overline{\mathbf{H}}^3$ is expressed with the dihedral angles $\alpha_{01}, \alpha_{12}, \alpha_{23}$, ($0 \leq \alpha_{ij} \leq \frac{\pi}{2}$) (Fig. 2) in the following form:*

$$vol(\mathcal{O}) = \frac{1}{4} \left\{ \mathcal{L}(\alpha_{01} + \theta) - \mathcal{L}(\alpha_{01} - \theta) + \mathcal{L}\left(\frac{\pi}{2} + \alpha_{12} - \theta\right) + \right. \\ \left. + \mathcal{L}\left(\frac{\pi}{2} - \alpha_{12} - \theta\right) + \mathcal{L}(\alpha_{23} + \theta) - \mathcal{L}(\alpha_{23} - \theta) + 2\mathcal{L}\left(\frac{\pi}{2} - \theta\right) \right\},$$

where $\theta \in [0, \frac{\pi}{2})$ is defined by the following formula:

$$\tan(\theta) = \frac{\sqrt{\cos^2 \alpha_{12} - \sin^2 \alpha_{01} \sin^2 \alpha_{23}}}{\cos \alpha_{01} \cos \alpha_{23}}$$

and where $\mathcal{L}(x) := -\int_0^x \log |2 \sin t| dt$ denotes the Lobachevsky function.

We obtain the volume formula for the asymptotical orthoschems if A_0 and A_3 are at the absolute i.e. $\alpha := \alpha_{01} = \frac{\pi}{2} - \alpha_{12} = \alpha_{23} = \theta$:

$$\text{vol}(\mathcal{O}_\infty) = \frac{1}{2}\mathcal{L}(\alpha). \quad (2.10)$$

J. Milnor's volume formula of a fully asymptotic tetrahedron $T(\alpha, \beta)$ with dihedral angles α, β, γ is determined by its division into orthoschems (see [14]):

$$\text{vol}(T(\alpha, \beta)) = \mathcal{L}(\alpha) + \mathcal{L}(\beta) + \mathcal{L}(\gamma), \quad \text{where } \alpha + \beta + \gamma = \pi. \quad (2.11)$$

As an easy consequence, we get the following

Lemma 2.4 *The volume formula $\text{vol}(T(\alpha))$ of a fully asymptotic plane symmetric tetrahedron $T(\alpha)$ can be derived by the duplication law:*

$$\text{vol}(T(\alpha)) = 2\mathcal{L}(\alpha) - \mathcal{L}(2\alpha) = -2\mathcal{L}(\alpha + \frac{\pi}{2}). \quad (2.12)$$

In [8] there are further results for the volumes of orthoschemes and simplices in higher dimensions.

3 Horoball packings for asymptotic tetrahedra

The aim of this section is to determine the optimal packing arrangement and its densities for the fully asymptotic tetrahedra in $\overline{\mathbb{H}}^3$.

We will make heavy use of the following Lemma (see also [17]):

Lemma 3.1 *Let B_1 and B_2 denote two horoballs with ideal centers C_1 and C_2 respectively. Take τ_1 and τ_2 to be two congruent trihedra, with vertices C_1 and C_2 . Assume that these horoballs $B_1(x)$ and $B_2(x)$ are tangent at point $I(x) \in C_1C_2$ and C_1C_2 is a common edge of the two trihedra τ_1 and τ_2 . We define the point of contact $I(0)$ such that the following equality holds for the volumes of horoball sectors:*

$$V(0) := 2\text{vol}(B_1(0) \cap \tau_1) = 2\text{vol}(B_2(0) \cap \tau_2).$$

If x denotes the hyperbolic distance between $I(0)$ and $I(x)$, then the function

$$V(x) := \text{vol}(B_1(x) \cap \tau_1) + \text{vol}(B_2(x) \cap \tau_2) = \frac{V(0)}{2}(e^{2x} + e^{-2x})$$

strictly increases as $x \rightarrow \pm\infty$.

We arbitrarily choose a fully asymptotic tetrahedron $T(\alpha) = E_0E_1E_2E_3$ with at least one plane of symmetry (see Section 1.4 and Fig. 3), and place the horoball centers at vertices E_0, \dots, E_3 . We vary the types of the horoballs so that they satisfy our constraints of non-overlap. The packing density is obtained by Definition 1.3. The dihedral angles of the above tetrahedron at the edges E_0E_1 , E_0E_2 , E_1E_3 , E_2E_3 are denoted by α and it is clear that the dihedral angles at the remaining edges E_0E_3 and E_1E_2 are $\pi - 2\alpha$.

We introduce a Euclidean projective coordinate system to the tetrahedron $T(\alpha)$ by the following coordinates of the vertices:

$$E_0 := (1, 0, \sqrt{1 - z^2}, z), \quad (-1 \leq z \leq 1); \quad E_1 = \left(1, \frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right);$$

$$E_2 = \left(1, -\frac{\sqrt{3}}{2}, 0, -\frac{1}{2}\right); \quad E_3 := (1, 0, 0, 1).$$

E_0 lies on the symmetry plane $x = 0$ of tetrahedron $T(\alpha)$ (see Fig. 3). In order to determine the optimal ball arrangement and its fully asymptotic tetrahedron first we consider the following situations:

1. In an optimally dense packing, the horoball packing must be locally stable, i.e. each ball is fixed by its neighboring horoballs or by the opposite face of the tetrahedron. Otherwise the density could be improved by blowing up at least one horoball until it touches a neighboring horoball or its opposite face of tetrahedron.
2. We fix the parameter z ($z \in (-1, 1) \setminus 0$) and blow up the horoballs B_i centered at the vertices of $T(\alpha)$ so that the volumes $\text{vol}(B_i \cap T(\alpha))$, ($i = 0, 1, 2, 3$) are equal until certain two neighbouring horoballs touch each other. In this situation we distinguish two cases for the varying vertex E_0 :
 - i.) If $0 < z < 1$ then the horoballs $B_3 - B_0$ and $B_1 - B_2$ touch each other the other horoballs do not.
 - ii.) If $-1 < z < 0$ then each of the horoball pairs $B_0 - B_1$, $B_0 - B_2$, $B_1 - B_3$, $B_2 - B_3$ have exactly one common point, respectively, and $B_3 - B_0$ and $B_1 - B_2$ do not touch each other.

If $z = 0$ the tetrahedron is regular $T(\alpha) = T_{reg}$, ($\alpha = \frac{\pi}{3}$) and each horoball touch all neighboring horoballs. This case is studied in details formerly in [9].

3. The volume $vol(T(\alpha))$ is determined by Lemma 2.4, formula (2.12), and by the above machinery of hyperbolic geometry we obtain the following formulas:

$$\begin{aligned} \cos(2\alpha) &= -\frac{1+2z}{z-2}, \quad (-1 < z < 1), \\ vol(T(\alpha)) &= -2\mathcal{L}\left[\frac{1}{2} \arccos\left(-\frac{1+2z}{z-2}\right) + \frac{\pi}{2}\right]. \end{aligned} \quad (3.1)$$

i/1: $(z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}])$

- a.) First, we define the tangent point $I(0) \in E_0E_3$ of horoballs $B_0(0)$ and $B_3(0)$ so that equalities hold for the volumes of all horoball sectors.

The volume sum of horoball sectors $B_i(0) \cap T(\alpha)$, $(i = 0, 1, 2, 3)$ in $T(\alpha)$ is

$$V(0) = \sum_{i=1}^4 (vol(B_i(0) \cap T(\alpha))) = 4vol(B_1(0) \cap T(\alpha)).$$

In the introduced coordinate system $B_1(0)$ and $B_2(0)$ touch each other at $(1, 0, 0, -\frac{1}{2})$. Consider the point $I(x)$ on the edge E_0E_3 where the horoballs

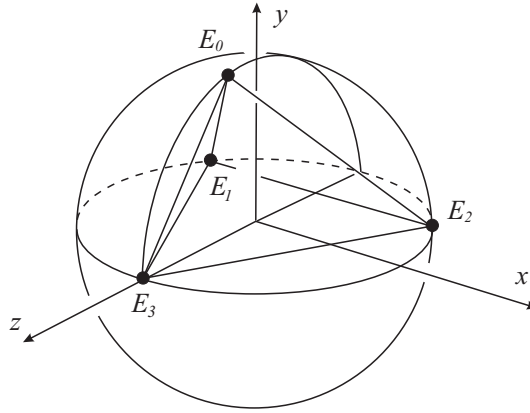


Figure 3:

B_i , $(i = 0, 3)$ are tangent. Here x denotes the hyperbolic distance between $I(0)$ and $I(x)$. Let $I(x_1) \in E_0E_3$ be such a point and parameter x_1 where

horoball $B_3(x_1)$ touches $B_2(0)$ and $B_1(0)$. This horoball arrangement is denoted by $\mathcal{B}_1(x_1(z))$. Function $V_1(x)$ to \mathcal{B}_1 is defined by:

$$V_1(x) := \frac{1}{2}V(0) + \frac{1}{4}V(0)e^{2x} + \frac{1}{4}V(0)e^{-2x}, \quad x \in (0, x_1(z)].$$

By Lemma 3.1 it follows that function $V_1(x)$ strictly increases as $I(x)$ ($x \in (0, x_1(z)]$) moves away from $I(0)$ along E_0E_3 . That means that the density function for any $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$, $(\frac{-2}{13} + \frac{6\sqrt{3}}{13} \approx 0.64556191)$

$$\delta(\mathcal{B}_1(x(z))) = \frac{V_1(x)}{\text{vol}(T(\alpha))}$$

attains its maximum at the endpoint x_1 . Such a configuration does not occur if $z \in (\frac{-2}{13} + \frac{6\sqrt{3}}{13}, 1]$.

To study the former density function $\delta(\mathcal{B}_1(x_1(z)))$ we have to compute the volume sum of the horoball sectors $V_1(x_1) = \Sigma_1^4(B_i \cap T(\alpha))$ belonging to the parameter x_1 . Of course the volume $V_1(x_1)$ and the volume of the tetrahedron $T(\alpha)$ depend on parameter z . First we calculate the six intersection points of the edges of $T(\alpha)$ and horoballs B_i ($i = 0, 1, 2, 3$), using the above projective coordinate system and the equations of the horospheres derived from (2.7). These intersection points on the edge E_iE_j is denoted by M_{ij}^1 ($i < j$), $i, j \in \{0, 1, 2, 3\}$.

$$\begin{aligned} M_{03}^1 &= \left(1, 0, -3\frac{\sqrt{1-z^2}}{2z-5}, -\frac{z+2}{2z-5}\right), \quad M_{13}^1 = \left(1, -\frac{\sqrt{3}}{4}, 0, \frac{1}{4}\right), \\ M_{23}^1 &= \left(1, \frac{\sqrt{3}}{4}, 0, \frac{1}{4}\right), \quad M_{12}^1 = \left(1, 0, 0, -\frac{1}{2}\right), \\ M_{02}^1 &= \left(1, \frac{\sqrt{3}(z+2)}{2(z+5)}, 3\frac{\sqrt{1-z^2}}{z+5}, \frac{5z-2}{2(z+5)}\right), \\ M_{01}^1 &= \left(1, -\frac{\sqrt{3}(z+2)}{2(z+5)}, 3\frac{\sqrt{1-z^2}}{z+5}, \frac{5z-2}{2(z+5)}\right). \end{aligned} \tag{3.2}$$

The volumes of horoball sectors (depending on parameter z) can be computed by Bolyai's formulas (2.8-9) and the volume of the tetrahedron $T(\alpha)$ is determined by formula (3.1). Finally, we obtain the density function $\delta(\mathcal{B}_1(x_1(z))) = \frac{V_1(x_1)}{\text{vol}(T(\alpha))}$ which depends only on parameter $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$. By careful computer analysis of the above density function we get

that the function is convex, it attains its maximum at the upper endpoint of the interval $(0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$. We have studied the above function with *Maple* using that the conditions of the Lobachevsky function are well known [22]. The graph of $\delta(\mathcal{B}_1(x_1(z)))$ can be seen in Fig. 4. Note, that the case

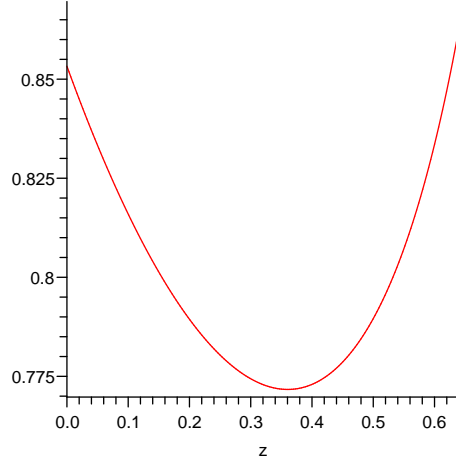


Figure 4:

$z = 0$ belongs to the optimal arrangement of Böröczky–Florian, its density is ≈ 0.85327609 . The density at the upper endpoint of the interval $(0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$ is larger than that, namely:

$$\delta\left(\mathcal{B}_1\left(x_1\left(\frac{-2}{13} + \frac{6\sqrt{3}}{13}\right)\right)\right) \approx 0.86767481, \quad (\alpha \approx 1.30899694). \quad (3.3)$$

- b.) We consider the former horoball arrangement $\mathcal{B}_1(x_1(z))$ for any $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$. We expand the above "larger horoball" $B_3(x_1)$ until it touches the opposite face $E_0E_1E_2$ of the tetrahedron while keeping the other horoballs tangent to it. The arising horoball arrangement are realizable for any $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$.

Similarly to the case a.) we consider a point $I(x)$ ($x \geq x_1$) (on the edge E_0E_3 where the horoballs B_i , ($i = 0, 3$) are tangent at point $I(x) \in E_0E_3$. Let $x - x_1$ denote the hyperbolic distance between $I(x_1)$ and $I(x)$. Furthermore, let $I(x_2) \in E_0E_3$ be such a point where horoball $B_3(x_2)$ of parameter

x_2 touches the face $E_0E_1E_2$. The "largest horoball" determines the configuration of all other horoballs and this horoball arrangement is denoted by $\mathcal{B}_2(x_2(z))$. Function $V_2(x)$ is defined as follows:

$$V_2(x) := \frac{1}{2}V(0)e^{-2(x-x_1)} + \frac{1}{4}V(0)e^{2x} + \frac{1}{4}V(0)e^{-2x}, \quad x \in [x_1, x_2].$$

The following Lemma is obtained by examining the above function:

Lemma 3.2 *The maxima of function $V_2(x)$ are realized to parameters x_1 or x_2 .*

By the above Lemma 3.2 it follows that it is sufficient to consider the volume function $V_2(x)$ and the density function $\delta(\mathcal{B}_2(x(z)))$ at the parameters x_1 and x_2 . The case x_1 was taken in a.) thus we have to consider only the case $x = x_2$.

The horoball $B_3(x_2)$ has to touch the side face $E_0E_1E_2$ of fully asymptotic tetrahedron $T(\alpha)$. Thus, it is passing through the foot point E'_3 perpendicularly dropped from the point E_3 on the plane $E_0E_1E_2$,

$$E'_3 = \left(1, 0, \frac{3\sqrt{(1-z^2)}(1+2z)}{(z+5)^2}, -\frac{7z^2+4z-2}{2z^2-4z-7}\right).$$

To examine the density function $\delta(\mathcal{B}_2(x_2(z)))$ we have to compute the sum of the volumes of the horoball sectors $V_2(x_2) = \sum_1^4 (B_i \cap T(\alpha))$ to the parameter x_2 . The volume $V_2(x_2)$ and the volume of the tetrahedron $T(\alpha)$ depend on parameter z . Similarly to the case a.) we have to determine the seven intersection points of the edges of $T(\alpha)$ and horoballs B_i ($i = 0, 1, 2, 3$). The intersection points on the edges E_0E_1 , E_0E_2 , E_0E_3 , E_3E_2 and E_3E_1 are denoted by M_{ij}^2 ($i < j$), $i, j \in \{0, 1, 2, 3\}$. On the edge E_1E_2 there are no touching point but there are two point of intersections $M_{12,2}^2$.

$$\begin{aligned} M_{03}^2 &= \left(1, 0, 18\frac{\sqrt{1-z^2}(1+z)}{4z^2+22z+22}, \frac{19z^2+22z+4}{4z^2+22z+22}\right), \\ M_{13}^2 &= \left(1, -\frac{6\sqrt{3}(z^2-1)}{11z^2-4z-16}, 0, -\frac{7z^2+4z-2}{11z^2-4z-16}\right), \\ M_{23}^2 &= \left(1, \frac{6\sqrt{3}(z^2-1)}{11z^2-4z-16}, 0, -\frac{7z^2+4z-2}{11z^2-4z-16}\right), \\ M_{12,2}^2 &= \left(1, \pm\frac{\sqrt{3}(13z^2+4z-8)}{2(11z^2-4z-16)}, 0, -\frac{1}{2}\right), \end{aligned} \tag{3.4}$$

$$M_{02}^2 = \left(1, \frac{2\sqrt{3}(z^2 - 1)}{4z^2 - z - 6}, -\frac{\sqrt{1 - z^2}(z + 2)}{4z^2 - z - 6}, -\frac{3z^2 + 2z - 2}{4z^2 - z - 6}\right),$$

$$M_{01}^2 = \left(1, -\frac{2\sqrt{3}(z^2 - 1)}{4z^2 - z - 6}, -\frac{\sqrt{1 - z^2}(z + 2)}{4z^2 - z - 6}, -\frac{3z^2 + 2z - 2}{4z^2 - z - 6}\right).$$

The volumes of horoball sectors (depending on parameter z) can be computed by Bolyai's formulas (2.8-9) and the volume of the tetrahedron $T(\alpha)$ is determined by formula (3.1). We get the density function $\delta(\mathcal{B}_2(x_2(z))) = \frac{V_2(x_2)}{\text{vol}(T(\alpha))}$ which depend only on parameter $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$. By careful investigation (see [22]) of the above density function we get that the function attains its maximum at the point $z = \frac{-2}{13} + \frac{6\sqrt{3}}{13}$. The graph of $\delta(\mathcal{B}_2(x_2(z)))$ is shown in Fig. 5. The density at the point $\frac{-2}{13} + \frac{6\sqrt{3}}{13}$ is larger than the

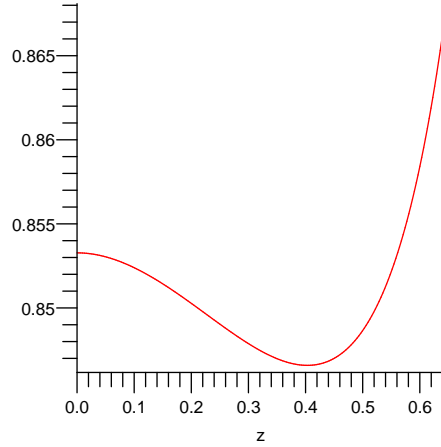


Figure 5:

Böröczky-Florian upper density, namely

$$\delta\left(\mathcal{B}_2\left(x_2\left(\frac{-2}{13} + \frac{6\sqrt{3}}{13}\right)\right)\right) \approx 0.86767481, \quad (\alpha \approx 1.30899694). \quad (3.5)$$

The horoball density of of Böröczky–Florian belongs to the parameter $z = 0$. To the parameter $z = \frac{-2}{13} + \frac{6\sqrt{3}}{13}$ we get the same horoball density as in case a.).

i/2: $(z \in (\frac{-2}{13} + \frac{6\sqrt{3}}{13}, 1])$

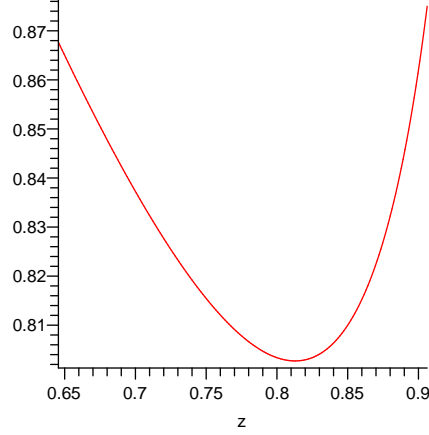


Figure 6:

- a.) $z \in \left(\frac{-2}{13} + \frac{6\sqrt{3}}{13}, z_3 \right]$, where $z_3 := \frac{4}{73} \sqrt{3} \sqrt{178} \cos(\frac{1}{3} \arctan(\frac{73}{8001} \sqrt{3} \sqrt{229})) - \frac{26}{73} \approx 0.9061774494$.

In the former case **i/1/b** we have already considered the ball arrangement $\mathcal{B}_1(x_1(z))$ for any $z \in (0, \frac{-2}{13} + \frac{6\sqrt{3}}{13}]$. There we have expanded the "larger horoball" $B_3(x_1)$ until it touches the opposite face $E_0E_1E_2$ of the tetrahedron while keeping other horoballs tangent to it. But if $z \in (\frac{-2}{13} + \frac{6\sqrt{3}}{13}, z_3]$ then we cannot apply this procedure. Namely, if the horoball $B_3(x_2)$ touches the opposite side face, then $B_3(x_2)$ does not touch $B_1(0)$ and $B_2(0)$. Then we expand the horoball B_2 until it touches $B_3(x_2)$ or the opposite side face. This means that the "larger horoball" $B_3(x_2)$ touches the opposite face $E_0E_1E_2$ at point E'_3 and the horoballs B_2, B_0 touch B_3 . Then B_1 touches only the horoball B_2 . This configuration occurs until the horoball B_2 is not tangent to its opposite face $E_3E_1E_0$.

This arrangement exactly exists if $z \in (\frac{-2}{13} + \frac{6\sqrt{3}}{13}, z_3]$. The horoball B_2 touches the opposite side face $E_0E_1E_3$ of $T(\alpha)$ at the point E'_2 :

$$E'_2 = \left(\frac{5z+7}{2(z+2)}, -\frac{\sqrt{3}(2z+1)}{2(z+2)}, \frac{3\sqrt{1-z^2}}{2(z+2)}, \frac{2z+1}{2(z+2)} \right). \quad (3.6)$$

Similarly to the case a.) the "largest horoball" determines the configuration of all other horoballs and this horoball arrangement is denoted by $\mathcal{B}_3(x_3(z))$.

The function $V_3(x_3(z))$ and related density function

$$\delta(\mathcal{B}_3(x_3(z))) = \frac{V_3(x_3(z))}{\text{vol}(T(\alpha))}$$

can be analysed in the same way as in **i/1/a** and **i/1/b** and we obtain that the convex density function attains its maximum at the above z_3 .

The graph of $\delta(\mathcal{B}_3(x_3, z))$, $z \in [\frac{-2}{13} + \frac{6\sqrt{3}}{13}, z_3]$ is shown in the Fig. 6. While Fig. 7 shows the larger horoballs $B_3, B_2 \in \mathcal{B}_3(x_3(z_3))$ with tetrahedron $T(\alpha)$ in the Beltrami-Cayley-Klein model. The density at the point $z = z_3$

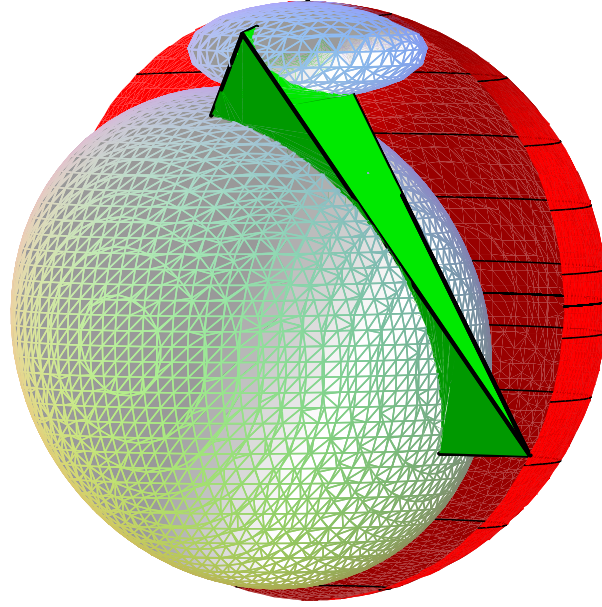


Figure 7: The optimal horoball packing arrangement related to a fully asymptotic tetrahedron

is larger than the above maxima

$$\delta(\mathcal{B}_3(x_3(z_3))) \approx 0.87499429, \quad (\alpha \approx 1.44340117). \quad (3.7)$$

b.) $z \in (z_3, 1)$

We consider such ball arrangements $\mathcal{B}_4(x_4(z))$ where the "larger horoballs" B_3 and B_2 touch their opposite faces $E_0E_1E_2$ and $E_0E_1E_3$, respectively. The horoballs B_0, B_3 and B_1, B_2 pairwise touch each other. This horoball

packing can be studied similarly to the above cases and we can analyse the density function $\delta(\mathcal{B}_4(x_4(z)))$ (see Fig. 8), not detailed further. We obtain, that the density function is strictly decreasing in the interval $(z_3, 1)$. Thus, there is no horoball arrangement with larger density than $\mathcal{B}_3(x_3(z_3))$ with density ≈ 0.87499429 (see (3.7)).

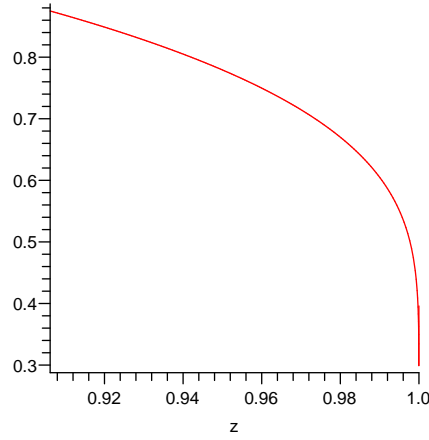


Figure 8:

Analogously to (i) we have investigated the cases (ii) $(-1 < z < 0)$, not detailed more, and we have obtained the following

Theorem 3.3 *The densest horoball arrangement, related to the generalized simplicial density function (Definition 1.3) belongs to the horoball arrangement $\mathcal{B}_3(x_3(z_3))$ (see (3.7) and Fig. 7) with density*

$$\delta(\mathcal{B}_3(x_3(z_3))) \approx 0.87499429.$$

Remark 3.4 *This packing seems not to have global extension to the entire hyperbolic space $\overline{\mathbb{H}}^3$, so that the same density occurs in each asymptotic tetrahedron. We plan to investigate this problem with E. Molnár and I. Prok by [12].*

Optimal sphere packings in other homogeneous Thurston geometries represent a class of open mathematical problems. For these non-Euclidean geometries only very few results are known [20], [21]. Detailed studies are the objective of ongoing research.

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